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Exponential trichotomy and homoclinic bifurcation with saddle-center equilibrium[☆]

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ABSTRACT

In this paper the bifurcation of a homoclinic orbit is studied for an ordinary differential equation with periodic perturbation. Exponential trichotomy theory with the method of Lyapunov–Schmidt is used to obtain some sufficient conditions to guarantee the existence of homoclinic solutions and periodic solutions for this problem. Some known results are extended.

1. Introduction

With the development of nonlinear science, an increasingly large number of papers have been devoted to the bifurcation problems of homoclinic orbits with nonhyperbolic equilibrium (see [1–5] and the references therein). The methods used in those papers work well in studying bifurcation problem under autonomous perturbation, but these methods have a large limitation in the discussion of the persistence and bifurcation problem for homoclinic orbits under periodic perturbation. In this paper we are interested in the problem of homoclinic bifurcation with nonautonomous perturbation, more precisely, we will consider the system

$$\begin{aligned} \dot{x} &= f(x, y) + \varepsilon g^x(x, y, t, \lambda, \varepsilon), \\ \dot{y} &= \varepsilon g^y(x, y, t, \lambda, \varepsilon) \end{aligned} \tag{1.1}$$

where $(x, y) \in R^n \times R^m$, $t \in R$, $\lambda \in R^k$, $0 < \varepsilon \ll 1$, g^x, g^y are 2π periodic with respect to t.

When $y = 0$, in [6–9] a functional analytic approach together with the method of Lyapunov–Schmidt is used to discuss the existence of homoclinic solutions and periodic solutions for (1.1) with a hyperbolic equilibrium. When $y \neq 0$, in [10], based on the *Poincaré* map, the bifurcation of subharmonic solutions and invariant tori are investigated. Inspired by [6–10], we will use exponential trichotomy theory with the method of Lyapunov–Schmidt to discuss the problem of homoclinic bifurcation for (1.1) when $y \neq 0$. By the Melnikov function, we give some sufficient conditions to guarantee the existence of homoclinic orbit and the existence of periodic orbit bifurcated from the homoclinic orbit. The main technical difference from the analysis of homoclinic orbits with a hyperbolic equilibrium is that in the nonhyperbolic situation the variation equation along the homoclinic orbit no longer has an exponential dichotomy but an exponential trichotomy. This requires a modified approach.

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2. Persistence of homoclinic orbit under perturbation

Consider the $C^r(r \geq 2)$ system (1.1) and the corresponding unperturbed system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= 0. \end{aligned} \tag{2.1}$$

We make the following assumptions:

(H_1) There exists $y_0 \in R^m$, such that system

$$\dot{x} = f(x, y) \tag{2.2}$$

has a hyperbolic equilibrium $x = x(y_0)$ and a homoclinic orbit $\Gamma = \{\gamma(t) : \gamma(\pm\infty) = x(y_0)\}$. The stable manifold W^s and unstable manifold W^u of $x(y_0)$ are n_1-dimensional and n_2-dimensional, respectively, with $n_1 + n_2 = n$. Moreover for any $\bar{p} \in \Gamma$,

$$\dim(W^s(x(y_0)) \cap W^u(x(y_0))) = \dim(T_{\bar{p}}W^s(x(y_0)) \cap T_{\bar{p}}W^u(x(y_0))) = 1.$$

Remark 2.1. From the assumption (H_1), it is easy to see that system (2.1) has an equilibrium $q(x(y_0), y_0)$, which possesses n_1-dimensional stable manifold $W^s(q)$, n_2-dimensional unstable manifold $W^u(q)$, and m-dimensional center manifold $W^c(q)$ respectively.

Suppose system (1.1) still has a invariant set $q(t, \varepsilon)$ under small perturbation, which satisfies

$$|q(t, \varepsilon) - q| = o(\varepsilon).$$

For convenience, we assume a transformation has already been made to move the equilibrium to the origin, then we can assume that

$$g^x(0, 0, t, \lambda, \varepsilon) = 0, \qquad g^y(0, 0, t, \lambda, \varepsilon) = 0.$$

Furthermore, we need the following assumption for (1.1)

(H_2) $D_y g^y(0, 0, t, \lambda, \varepsilon) = \operatorname{diag}(C_1(\varepsilon), C_2(\varepsilon))$, $g^y(x, y, t, \lambda, \varepsilon) = 0$ is odd with respect to t. $C_1(\varepsilon)$ is a $m_1 \times m_1$ matrix, $C_2(\varepsilon)$ is a $m_2 \times m_2$ matrix, $m_1 + m_2 = m$, $\operatorname{Re}\sigma(C_1(\varepsilon)) > 0$, $\operatorname{Re}\sigma(C_2(\varepsilon)) < 0$.

Consider the linear variational system of $(1.1)|_{\varepsilon=0}$

$$\dot{U} = A(t)U \tag{2.3}$$

and its adjoint system

$$\dot{V} = -A^*(t)V \tag{2.4}$$

where $A(t) = \partial(f + \varepsilon g^x, \varepsilon g^y)/\partial(x, y)(\gamma(t), 0)|_{\varepsilon=0}$, the sign "$*$" denotes the transposition.

Take $p = (\gamma(t), 0)$. Based on (H_1) and $\dot{y} = 0$, the tangent space corresponding to (2.3) can be decomposed into

$$T_pR^{n+m} = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \tag{2.5}$$

which satisfy

$$\begin{aligned} T_1 &= T_pW^s(0) \bigcap T_pW^u(0), \\ T_2 &= T_pW^s(0)/[T_pW^s(0) \bigcap T_pW^u(0)], \\ T_3 &= T_pW^u(0)/[T_pW^s(0) \bigcap T_pW^u(0)], \\ T_4 &= T_pW^c(0), \\ T_5 &= [T_pW^s(0) + T_pW^u(0)]^c \bigcap [T_pW^c(0)]^c. \end{aligned}$$

From (H_1) and the above decomposition, system (2.3) has exponentially bounded solution $\phi(t)$, and system (2.4) has exponentially bounded solution $\psi(t)$, such that $\phi(t), \psi(t) \to 0$ exponentially as $t \to \pm\infty$. Also we have $\langle\phi, \psi\rangle = 0$, and $T_1 = \operatorname{span}\{\phi(t)\}$, $T_5 = \operatorname{span}\{\psi(t)\}$.

Now we choose a fundamental solution matrix $X(t, t_0)$ of (2.3) satisfying $X(t_0, t_0) = Id$, then $Y(t, t_0) = X^{-1*}(t, t_0)$ is a fundamental solution matrix of (2.4).

Definition 2.1. We say (2.3) has an exponential trichotomy in J if there exist projections $P_c(t)$, $P_s(t)$ and $P_u(t) = I - P_c - P_s$, $t \in J$, and there are constants $K \geq 1$, and $\alpha \gg \sigma > 0$ such that for $t, s \in J$,

$$
\begin{aligned}
&X(t,s)P_v(s) = P_v(t)X(t,s), \quad t \geq s,\ v = c, u, s,\\
&|X(t,s)P_c(s)| \leq K\mathrm{e}^{\sigma|t-s|},\\
&|X(t,s)P_s(s)| \leq K\mathrm{e}^{-\alpha(t-s)}, \quad t \geq s,\\
&|X(s,t)P_u(t)| \leq K\mathrm{e}^{-\alpha(t-s)}, \quad t \geq s.
\end{aligned}
$$

Remark 2.2. If $P_c(t) = 0$, then (2.3) is said to have an exponential dichotomy in J.

Similar to [11], we get the following result

Lemma 2.2. *If* (H_1) *holds, then* (2.3) *and* (2.4) *have an exponential trichotomy both in* R^+ *and* R^- *with the same constants* K, α, σ, *and the corresponding projections are* $P_s^i(t)$, $P_c^i(t)$, $P_u^i(t)$ *and* $P_s^{i*}(t)$, $P_c^{i*}(t)$, $P_u^{i*}(t)$, $i = +, -$, *respectively. Moreover*

$$
\begin{aligned}
&RP_s^+ = T_pW^s, \qquad RP_u^- = T_pW^u,\\
&RP_c^- = RP_c^+ = T_pW^c,\\
&R(P_s^+ + P_c^+) = T_pW^{cs}, \qquad R(P_c^- + P_u^-) = T_pW^{cu}.
\end{aligned}
$$

Remark 2.3. From (2.5) and Lemma 2.2, it is obvious that

$$
\begin{aligned}
&\dim(RP_s^+(t) \cap RP_u^-(t)) = 1,\\
&\dim(RP_u^{+*}(t) \cap RP_s^{-*}(t)) = 1.
\end{aligned}
$$

Denote

$$
\begin{aligned}
&E(b,J) = \{x \in C^0 : \sup_{t\in J}\{|x(t)|\mathrm{e}^{b|t|}\} < \infty\},\\
&E(b,r,J) = \{x \in C^r : x, \ldots, x^{(r)} \in E(b,J)\}, \quad r \in N.
\end{aligned}
$$

Then $E(b,J)$ and $E(b,r,J)$ are Banach spaces with norms $\|x\|_0 = \sup_{t\in J}\{|x(t)|\mathrm{e}^{b|t|}\}$, $\|x\|_r = \sum_{k=0}^{r}\|x^{(k)}\|_0$, respectively.

Now we consider the nonhomogeneous equation

$$\dot{z} = A(t)z + h(t). \tag{2.6}$$

Denote

$$Z = \left\{h(t) \in E(-\sigma, R) : \int_{-\infty}^{+\infty} \psi^*(t)h(t)\mathrm{d}t = 0,\ \forall\psi \in E(\alpha, 1, R),\ \dot{\psi} = -A^*(t)\psi\right\}. \tag{2.7}$$

Lemma 2.3. *Suppose that* (H_1) *is valid, and* $h(t) \in Z$, *then system* (2.6) *has bounded solution* $z(t)$ *in* $E(-\sigma, 1, R)$.

Proof. From the discussion above, system (2.4) has exponentially bounded solution $\psi(t) \in E(\alpha, 1, R)$, then there exists an vector $\eta \in R^{n+m}$ such that

$$\psi(t) = \begin{cases} Y(t,0)P_u^{+*}(0)\eta, & t \geq 0,\\ Y(t,0)P_s^{-*}(0)\eta, & t \geq 0.\end{cases} \tag{2.8}$$

It is obvious that

$$P_u^{+*}(0)\eta = P_s^{-*}(0)\eta$$

which is equivalent to

$$(P_s^{+*}(0) + P_c^{+*}(0))\eta = (P_u^{-*}(0) + P_c^{-*}(0))\eta.$$

Then we have

$$\eta^*[P_s^+(0) + P_c^+(0) - P_u^-(0) - P_c^-(0)] = 0. \tag{2.9}$$

Substitute (2.8) into (2.7) and notice that $Y(t,s) = X^*(s,t)$, we get

$$\eta^*\left[\int_0^{\infty} P_u^+(0)X(0,t)h(t)\mathrm{d}t + \int_{-\infty}^{0} P_s^-(0)X(0,t)h(t)\right] = 0. \tag{2.10}$$

Due to (2.9) and (2.10), there exists a vector $\xi \in R^{n+m}$ satisfying

$$[P_s^+(0) + P_c^+(0) - P_u^-(0) - P_c^-(0)]\xi = \int_0^{\infty} P_u^+(0)X(0,t)h(t)\mathrm{d}t + \int_{-\infty}^{0} P_s^-(0)X(0,t)h(t). \tag{2.11}$$

It follows from the Definition 2.1 that

$$P_j^i(0)X(0,t) = X(0,t)P_j^i(t), \quad i = +, -;\ j = s, u.$$

Then we can obtain

$$[P_s^+(0) + P_c^+(0)]\xi - \int_0^{\infty} P_u^+(0)X(0,t)h(t)\mathrm{d}t = [P_u^-(0) + P_c^-(0)]\xi + \int_{-\infty}^{0} P_s^-(0)X(0,t)h(t). \tag{2.12}$$

By variation of constants formula, we can get a function $z(t) \in E(-\sigma, 1, R)$ such that

$$z(t) = X(t,0)[P_s^+(0) + P_c^+(0)]\xi + \int_0^t X(t,s)[P_s^+(s) + P_c^+(s)]h(s)\mathrm{d}s - \int_t^{\infty} X(t,s)P_u^+(s)h(s)\mathrm{d}s, \quad t \geq 0,$$
$$z(t) = X(t,0)[P_u^-(0) + P_c^-(0)]\xi + \int_0^t X(t,s)[P_u^-(s) + P_c^-(s)]h(s)\mathrm{d}s + \int_{-\infty}^t X(t,s)P_s^-(s)h(s)\mathrm{d}s, \quad t \geq 0.$$

It is easy to see that $z(t)$ is the bounded solution of (2.7) in $E(-\sigma, 1, R)$. □

Denote a projection π as follows

$$\pi : E(-\sigma, R) \to E(-\sigma, R)$$
$$\pi z(t) = \psi(t)N^{-1}\int_{-\infty}^{+\infty} \psi^*(t)z(t)\mathrm{d}t$$

where $N = \int_{-\infty}^{+\infty} \psi^*(t)\psi(t)\mathrm{d}t$. Since $\mathrm{Im}(I - \pi) \in Z$, it follows from Lemma 2.3 that

$$\dot{z} = A(t)z + (I - \pi)h(t) \tag{2.13}$$

has a unique bounded solution $z(t)$ in $E(-\sigma, 1, R)$.

Now we consider the perturbed system (1.1). For convenience, we rewrite it as follows

$$\dot{z} = f(z) + \varepsilon h(z, t, \lambda, \varepsilon) \tag{2.14}$$

where $(x, y)^* = z, f(z) = (f(x, y), 0)^*, h = (g^x, g^y)^*$.

We now make the change of variable

$$w(t) = z(t + t_0) - s(t)$$

where $w(t) = (x, y)^*, s(t) = (\gamma(t), 0)^*$. Then system (2.14) can be changed into

$$\dot{w}(t) = A(t)w + G(w, t + t_0, \lambda, \varepsilon) \tag{2.15}$$

where

$$G(w, t + t_0, \lambda, \varepsilon) = f(w + s(t)) - f(s(t)) - A(t)w + \varepsilon h(w + s(t), t + t_0, \lambda, \varepsilon).$$

Based on the method of Lyapunov–Schmidt, (2.15) is equivalent to the following system

$$\dot{w}(t) = A(t)w + (I - \pi)G(w, t + t_0, \lambda, \varepsilon), \tag{2.16}$$
$$\pi G(w, t + t_0, \lambda, \varepsilon) = 0. \tag{2.17}$$

Since $w(t) \in E(-\sigma, R)$, it is easy to see

$$G : E(-\sigma, R) \times R \times R^k \times R \to E(-\sigma, R).$$

Also for all bounded solution $\psi(t)$ of (2.4) in $E(\alpha, 1, R)$, we have

$$\int_{-\infty}^{+\infty} \psi^*(t)(I - \pi)G(w, t + t_0, \lambda, \varepsilon)\mathrm{d}t = 0. \tag{2.18}$$

Owing to Lemma 2.3, system (2.16) has a unique nontrivial bounded solution $w = w(t, \lambda, \varepsilon)$ in $E(-\sigma, \lambda, \varepsilon)$ satisfying $w(t, \lambda, 0) = 0$. Then if (2.17) holds, namely

$$\int_{-\infty}^{+\infty} \psi^*(t)G(w, t + t_0, \lambda, \varepsilon)\mathrm{d}t = 0, \tag{2.19}$$

it follows that system (2.15) has a unique nontrivial bounded solution $w = w(t, \lambda, \varepsilon)$ in $E(-\sigma, \lambda, \varepsilon)$.

Let

$$H(t_0,\lambda,\varepsilon)=\int_{-\infty}^{+\infty}\psi^*(t)G(w,t+t_0,\lambda,\varepsilon)\mathrm{d}t. \tag{2.20}$$

Based on the definition of G, it is easy to see that $G(0,t+t_0,\lambda,0)=0$. Since $w(t,\lambda,0)=0$, we get $H(t_0,\lambda,0)=0$, then by simple calculation we obtain

$$H_\varepsilon=\int_{-\infty}^{+\infty}\psi^*(t)h(s(t),t+t_0,\lambda,0)\mathrm{d}t.$$

Let $M(t_0,\lambda)=H_\varepsilon(t_0,\lambda,0)$, (2.20) can be expressed into the following form

$$H(t_0,\lambda,\varepsilon)=\varepsilon[M(t_0,\lambda)+o(\varepsilon)].$$

From (2.14), we know that

$$M(t_0,\lambda)=\begin{bmatrix}\displaystyle\int_{-\infty}^{+\infty}\psi^*(t)g^x(\gamma(t),t+t_0,\lambda,0)\mathrm{d}t\\ \displaystyle\int_{-\infty}^{+\infty}\psi^*(t)g^y(\gamma(t),t+t_0,\lambda,0)\mathrm{d}t\end{bmatrix}.$$

By the implicit function theorem, we have the following results

Theorem 2.4. *Suppose that* (H_1) *holds. If there exists* $\tilde{t}_0$, $\tilde{\lambda}_0$ *such that* $M(\tilde{t}_0,\tilde{\lambda}_0)=0$, $M_\lambda(\tilde{t}_0,\tilde{\lambda}_0)\neq 0$, *then there exists a parameter surface* $\lambda=\lambda(t_0,\varepsilon)$ *satisfying* $\lambda(\tilde{t}_0,0)=\tilde{\lambda}_0$, *such that system* (2.15) *has a unique bounded solution* $w(t,\lambda)$ *in* $E(-\sigma,1,R)$ *for* $\lambda=\lambda(t_0,\varepsilon)$ *and* $0<\varepsilon\ll 1$, *namely, system* (1.1) *has a* 1*-homoclinic orbit* $\Gamma(t,\varepsilon)=w(t)+s(t)$ *satisfying*

$$|\Gamma(t,\varepsilon)-(\gamma(t),0)|=o(|\varepsilon|).$$

3. Existence of periodic solution

In this section we consider the existence of periodic solution bifurcated from homoclinic solution under perturbation. First we consider the following system

$$\dot{z}=A(t)z+h(t). \tag{3.0}$$

Let $T(\varepsilon)=\pi[\frac{1}{\varepsilon}]$. It is obvious that $T(\varepsilon)\to\infty$ as $\varepsilon\to 0$. For $t\in[-T(\varepsilon),T(\varepsilon)]$, we use variation of constant and exponential trichotomy to construct the following two solutions

$$\begin{aligned}z_1(t)&=X(t,0)[P_s^+(0)+P_c^+(0)]\xi_1+\int_0^t X(t,s)[P_s^+(s)+P_c^+(s)]h(s)\mathrm{d}s\\&\quad+X(t,T(\varepsilon))P_u^+(T(\varepsilon))\eta_1-\int_t^{T(\varepsilon)}X(t,s)P_u^+(s)h(s)\mathrm{d}s,\quad t\geq 0,\\z_2(t)&=X(t,0)[P_u^-(0)+P_c^-(0)]\xi_2+\int_0^t X(t,s)[P_u^-(s)+P_c^-(s)]h(s)\mathrm{d}s\\&\quad+X(t,-T(\varepsilon))P_s^-(-T(\varepsilon))\eta_2+\int_{-T(\varepsilon)}^t X(t,s)P_s^-(s)h(s)\mathrm{d}s,\quad t\geq 0.\end{aligned}$$

Denote new sets as follows

$$E_\varepsilon=C^1([-T(\varepsilon),T(\varepsilon)],R^{n+m})\cap E(-\sigma,R),$$

$$E_\varepsilon^h=\left\{h\in E_\varepsilon,\int_{-T(\varepsilon)}^{T(\varepsilon)}\psi^*(t)h(t)\mathrm{d}t+\psi^*(-T(\varepsilon))\eta_2-\psi^*(T(\varepsilon))\eta_1=0\right\}.$$

Owing to the definition of $z_1(t)$ and $z_2(t)$, if we can prove that

$$z_1(0)=z_2(0),\qquad z_1(T(\varepsilon))=z_2(-T(\varepsilon)), \tag{3.1}$$

then we can get a periodic solution in $[-T(\varepsilon),T(\varepsilon)]$ for $h(t)\in E_\varepsilon$ such that

$$z(t)=\begin{cases}z_1(t), & t\geq 0,\\ z_2(t), & t\leq 0.\end{cases}$$

Next we will prove that ξ_i, η_i $i=1,2$ can be selected suitably such that the two equalities in (3.1) hold. First we consider the variational equation (2.3) for $t\in[0,+\infty]$. Suppose a linear transformation has already been made such that the coefficient matrix of linear variational system of (2.1) in origin is given by $\mathrm{diag}(A_1, A_2, 0)$, where $\mathrm{Re}\,\sigma(A_1) < 0$, $\mathrm{Re}\,\sigma(A_2) > 0$. Take $\gamma(t) = (x_1(t), 0) \in T_{\gamma(t)}W^s$ for t large enough, it is easy to see

$$A(t) = \begin{pmatrix} A_1 + o(x_1(t)) & O(x_1(t)) & O(x_1(t)) \\ 0 & A_2 + o(x_1(t)) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $x_1(t) \to 0$ exponentially as $t \to +\infty$, then $A(+\infty) = A_0 = \mathrm{diag}(A_1, A_2, 0)$. Similarly, we can get $A(-\infty) = A_0$. The next lemma and its proof are a slightly extension of the Lemma 1 in [8].

Lemma 3.1. *Given $A(t)$, A_0 and its Jordan form $\bar{A}_0$, there exists a constant matrix C and a constant $a > 0$, such that*

$$\lim_{t\to\pm\infty} X(t, t_0)\mathrm{e}^{\mp at\bar{A}_0} = C.$$

In order to give a precise analysis, we decompose the projections $P_s^+, P_u^+, P_s^-, P_u^-$, respectively, into the following

$$P_{ss} + P_{us} = P_s^+, \qquad P_{ss} + P_{su} = P_u^-, \qquad P_{us} + P_{uu} = P_s^-, \qquad P_{su} + P_{uu} = P_u^+$$

such that

$$RP_{ss} = \mathrm{span}\{\phi(t)\}, \qquad RP_{uu} = \mathrm{span}\{\psi(t)\}.$$

From (2.5), we can take $z_i(0)$ satisfying

$$P_{ss}z_i(0) = 0, \qquad P_c^+ z_1(0) = 0, \qquad P_c^- z_2(0) = 0. \tag{3.2}$$

Then from the expression of $z_1(t), z_2(t)$, we know that $z_1(0) = z_2(0)$ is equivalent to

$$P_{us}\xi_1 - X(0, -T(\varepsilon))P_{us}\eta_2 - \int_{-T(\varepsilon)}^{0} X(0, s)P_{us}h(s)\mathrm{d}s = 0, \tag{3.3}$$

$$P_{su}\xi_2 - X(0, T(\varepsilon))P_{su}\eta_1 - \int_{T(\varepsilon)}^{0} X(0, s)P_{su}h(s)\mathrm{d}s = 0, \tag{3.4}$$

$$P_{uu}X(0, -T(\varepsilon))\eta_2 - P_{uu}X(0, T(\varepsilon))\eta_1 + \int_{-T(\varepsilon)}^{T(\varepsilon)} P_{uu}X(0, s)h(s)\mathrm{d}s = 0. \tag{3.5}$$

We solve (3.3) and (3.4) for ξ_1, ξ_2 respectively and substitute them into the equation

$$z_1(T(\varepsilon)) = z_2(-T(\varepsilon))$$

and rearrange terms to get the equation

$$\begin{aligned} &X(T(\varepsilon), 0)\left[X(0, -T(\varepsilon))P_{us}\eta_2 + \int_{-T(\varepsilon)}^{0} X(0, s)P_{us}h(s)\mathrm{d}s\right] + \int_{0}^{T(\varepsilon)} X(T(\varepsilon), s)(P_s^+ + P_c^+)h(s)\mathrm{d}s + (P_{su} + P_{uu})\eta_1 \\ &\quad = X(-T(\varepsilon), 0)\left[X(0, T(\varepsilon))P_{su}\eta_1 + \int_{T(\varepsilon)}^{0} X(0, s)P_{su}h(s)\mathrm{d}s\right] + \int_{0}^{-T(\varepsilon)} X(-T(\varepsilon), s)(P_u^- + P_c^-)h(s)\mathrm{d}s + (P_{us} + P_{uu})\eta_2 \end{aligned}$$

which can be simplified into

$$\begin{aligned} &[-X(T(\varepsilon), 0)P_{su} + P_{uu}X(0, T(\varepsilon)) + X(-T(\varepsilon), 0)P_{su}X(0, T(\varepsilon))]\eta_1 \\ &\quad + [-X(T(\varepsilon), 0)P_{us}X(0, -T(\varepsilon)) + X(-T(\varepsilon), 0)(P_{us} + P_{uu})X(0, -T(\varepsilon))]\eta_2 \\ &\quad = \int_{0}^{T(\varepsilon)} X(T(\varepsilon), s)(P_{ss} + P_{us} + P_c^+)h(s)\mathrm{d}s + \int_{0}^{T(\varepsilon)} X(-T(\varepsilon), s)P_{su}h(s)\mathrm{d}s \\ &\quad + \int_{-T(\varepsilon)}^{0} X(T(\varepsilon), s)P_{us}h(s)\mathrm{d}s + \int_{-T(\varepsilon)}^{0} X(-T(\varepsilon), s)(P_{ss} + P_{su} + P_c^-)h(s)\mathrm{d}s \equiv L(h, \varepsilon). \end{aligned} \tag{3.6}$$

Notice that $\dot{y} \equiv 0$ in (2.1). Based on assumption (H_2) and the construction of fundamental solution matrix $X(t, s)$ in [2], we know that

$$\int_{0}^{T(\varepsilon)} X(T(\varepsilon), s)P_c^+ h(s)\mathrm{d}s = \int_{-T(\varepsilon)}^{0} X(-T(\varepsilon), s)P_c^- h(s)\mathrm{d}s.$$

Also based on Lemma 3.1 and [8], we have

$$\sup_{t\le 0}\left|X(-T(\varepsilon),0)(P_{us}+P_{uu})X(0,-T(\varepsilon))-C\begin{pmatrix}I_{n_1\times n_2}&&\\&O_{n_2\times n_2}&\\&&O_{m\times m}\end{pmatrix}C^{-1}\right|e^{-2Mt}<\infty,$$

$$\sup_{t\ge 0}\left|X(T(\varepsilon),0)(P_{su}+P_{uu})X(0,T(\varepsilon))-C\begin{pmatrix}O_{n_1\times n_2}&&\\&I_{n_2\times n_2}&\\&&O_{m\times m}\end{pmatrix}C^{-1}\right|e^{2Mt}<\infty.$$

Then (3.6) can be changed into

$$\begin{aligned}&\left\{\begin{pmatrix}I_{n_1\times n_1}&&\\&0&\\&&0\end{pmatrix}+C^{-1}[X(-T(\varepsilon),0)(P_{us}+P_{uu})X(0,-T(\varepsilon))-C\begin{pmatrix}I_{n_1\times n_1}&&\\&0&\\&&0\end{pmatrix}C^{-1}\right.\\&\quad\left.-\ X(T(\varepsilon),0)P_{us}X(0,-T(\varepsilon))]C\right\}C^{-1}\eta_2+\left\{-\begin{pmatrix}0&&\\&I_{n_2\times n_2}&\\&&0\end{pmatrix}-C^{-1}[X(T(\varepsilon),0)(P_{su}+P_{uu})\right.\\&\quad\left.\times\ X(0,T(\varepsilon))-C\begin{pmatrix}0&&\\&I_{n_2\times n_2}&\\&&0\end{pmatrix}C^{-1}+X(-T(\varepsilon),0)P_{su}X(0,T(\varepsilon))]C\right\}C^{-1}\eta_1=C^{-1}L(h,\varepsilon).\end{aligned}$$

Owing to Lemma 3.1, we have

$$\left[\begin{pmatrix}I_{n_1\times n_1}&&\\&0&\\&&0\end{pmatrix}+P_1(\varepsilon)\right]C^{-1}\eta_2+\left[-\begin{pmatrix}0&&\\&I_{n_2\times n_2}&\\&&0\end{pmatrix}+P_2(\varepsilon)\right]C^{-1}\eta_1=C^{-1}L(h,\varepsilon)\tag{3.7}$$

where $P_1(\varepsilon)=o(e^{-k_1T(\varepsilon)})$, $P_2(\varepsilon)=o(e^{-k_2T(\varepsilon)})$, k_1,k_2 are positive constants. If we take

$$C^{-1}\eta_1=-\begin{pmatrix}0\\z_2\\0\end{pmatrix},\qquad C^{-1}\eta_2=\begin{pmatrix}z_1\\0\\0\end{pmatrix},$$

then (3.7) becomes

$$\left[\begin{pmatrix}I_{n_1\times n_1}&&\\&I_{n_2\times n_2}&\\&&0\end{pmatrix}+P(\varepsilon)\right]\eta=C^{-1}L(h,\varepsilon)\tag{3.8}$$

where $\eta=(z_1,z_2,0)^*$, $P(\varepsilon)=O(e^{-kT(\varepsilon)})$, $k=min\{k_1,k_2\}$. Take $\varepsilon_0>0$ small enough, then for $0<\varepsilon\le\varepsilon_0$, (3.8) has solution $\eta(h)$ for given $h(t)\in E_\varepsilon$, thus we get η_1,η_2 such that

$$z_1(T(\varepsilon))=z_2(-T(\varepsilon)).$$

We have now solved (3.3), (3.4) and $z_1(T(\varepsilon))=z_2(-T(\varepsilon))$. It is obvious that if equality (3.5) holds for given η_1,η_2, namely, $h(t)\in E^h_\varepsilon$, we have the following results

Lemma 3.2. *If $h(t)\in E^h_\varepsilon$, and assumption* (H$_1$), (H$_2$) *hold, system* (3.0) *has a $2T(\varepsilon)$-periodic solution $z_\varepsilon(t)$ for $0<\varepsilon\ll 1$.*

Now we discuss the existence of periodic solutions to (1.1), and seek periodic solutions of very large period which in some sense are near $\gamma(t)$.

For convenience, we consider (2.14) and make the change of variable

$$w(t)=z(t+t_0)-s(t)-\frac{1}{2T(\varepsilon)}e(\varepsilon)t,\tag{3.9}$$

where $e(\varepsilon)=s(-T(\varepsilon))-s(T(\varepsilon))$. Now (2.14) becomes

$$\dot w(t)=A(t)w+Q(w,t+t_0,\lambda,\varepsilon),\tag{3.10}$$

where

$$Q(w,t+t_0,\lambda,\varepsilon)=f\left(w(t)+s(t)+\frac{1}{2T(\varepsilon)}e(\varepsilon)t\right)-f(s(t))-A(t)w-\frac{1}{2T(\varepsilon)}e(\varepsilon)+\varepsilon h.$$

From (3.9), we know that system (2.14) has $2T(\varepsilon)$-periodic solution if and only if (3.10) has $2T(\varepsilon)$-periodic solution $w(t)$ satisfying $w(-T(\varepsilon))=w(T(\varepsilon))$. The condition for this is $Q\in E^h_\varepsilon$ which is equivalent to the following bifurcation equation

$$\int_{-T(\varepsilon)}^{T(\varepsilon)}\psi^*(t)Q(w,t+t_0,\lambda,\varepsilon)dt+\psi^*(-T(\varepsilon))\eta_2-\psi^*(T(\varepsilon))\eta_1=0.$$

Denote

$$\tilde{H}(t_0, \lambda, \varepsilon) = \int_{-T(\varepsilon)}^{T(\varepsilon)} \psi^*(t)Q(w, t + t_0, \lambda, \varepsilon)\mathrm{d}t + \psi^*(-T(\varepsilon))\eta_2 - \psi^*(T(\varepsilon))\eta_1. \tag{3.11}$$

We obtain the following Theorem

Theorem 3.3. *If* (H_1), (H_2) *hold, suppose we have a point* $(\tilde{t}_0, \tilde{\lambda}_0)$ *such that*

$$\tilde{H}(\tilde{t}_0, \tilde{\lambda}_0, 0) = 0, \qquad \tilde{H}_\lambda(\tilde{t}_0, \tilde{\lambda}_0, 0) \neq 0,$$

then there exists a parameter surface $\lambda = \lambda(t_0, \varepsilon)$ *satisfying* $\lambda(\tilde{t}_0, 0) = \tilde{\lambda}_0$, *such that* (1.1) *has a* $2T(\varepsilon)$*-periodic solution for* $\lambda = \lambda(t_0, \varepsilon)$, $0 < \varepsilon \ll 1$.

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